HW 1: Algebra/Intro Closes Tonight!
HW 2: 2.2 Separable eqns, slope fields

2.2: Separable Differential Equations

Entry Task: *(Motivation)*
Implicitly differentiate $x^2 + y^3 = 8$ and solve for $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[x^2 + y^3 = 8\right]$$

$$2x + 3y^2\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{3y^2}$$

$\boxed{\text{NOTE}}$:

$$\int 2x\,dx = x^2 + C_1$$

$$\int 3y^2\,dy = y^3 + C_2$$

Idea: Separate... integrate both sides.

Entry Task continued:
Find the *explicit* solution for

$$\frac{dy}{dx} = \frac{-2x}{3y^2}$$

with $y(0) = 2$.

$$3y^2\frac{dy}{dx} = -2x$$

$$\int 3y^2\,dy = \int -2x\,dx$$

$$y^3 + C_2 = -x^2 + C_1$$

$$\Rightarrow x^2 + y^3 = \underbrace{C_1 - C_2}_{\text{A CONSTANT}} = C$$

$y(0)=2 \Rightarrow \begin{matrix} x=0 \\ y=2 \end{matrix} \Rightarrow 0^2 + 2^3 = C$

$x^2 + y^3 = 8$ ← implicit sol'n

$\boxed{y = (8-x^2)^{1/3}}$ ← explicit sol'n

Separable Differential Equations

A **separable** differential equation can be written as:

$$\frac{dy}{dx} = f(x)g(y).$$

(or $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ or $\frac{dy}{dx} = \frac{g(y)}{f(x)}$.)

Example: Find the explicit solution to

$$\frac{dy}{dx} = -3xy$$

with $y(0) = 4$.

LINEAR

$y^1 \leftarrow$ 1st power only

$$\int \frac{1}{y}\, dy = \int -3x\, dx$$

$$\ln|y| = -\frac{3}{2}x^2 + C_1$$

$$|y| = e^{\left(-\frac{3}{2}x^2 + C_1\right)}$$

$$y = \underbrace{\pm e^{C_1}}_{C_2} e^{-\frac{3}{2}x^2}$$

$$\boxed{y = C_2 e^{-\frac{3}{2}x^2}}$$

$y(0) = 4 \Rightarrow \quad 4 = C_2 e^0 \Rightarrow C_2 = 4$

$$\boxed{y = 4e^{-\frac{3}{2}x^2}}$$

ASIDE: $4 = \pm e^{C_1}$

$C_1 = \ln(4)$

NO NEED TO DO THIS

JUST OBSERVE $C_1 \neq C_2$.

You do: Find the explicit solution to

$$\frac{dy}{dx} = 2xy^2$$

NONLINEAR

with $y(2) = \frac{1}{5}$.

$$\int \frac{1}{y^2}\,dy = \int 2x\,dx$$

$$-\frac{1}{y} = x^2 + C_1$$

$$\boxed{y = -\frac{1}{(x^2 + C_1)}}$$

$y(2) = 1/5$

$$\Rightarrow \quad \frac{1}{5} = -\frac{1}{4 + C_1}$$

$$\Rightarrow \quad 4 + C_1 = -5$$

$$C_1 = -9$$

$$\boxed{y = \frac{-1}{x^2 - 9}}$$

What if the initial condition was $y(2) = 0$?

$$0 = -\frac{1}{2^2 + C_1} \Rightarrow 0 = -1$$

WHAT?!?

SOMETHING IS WRONG WITH THE INITIAL CONDITIONS?!?

OR

THERE IS NO SOL'N IN THE FORM SPECIFIED?

WE WILL DISCUSS EQUILIBRIUM SOLUTIONS LATER. THESE ARE CONSTANT SOLUTIONS WHEN $\frac{dy}{dx}$ IS ALWAYS ZERO.

NOTE IF $y(t) = 0$, for all t THEN $y(t)$ IS A SOL'N!

AND $y(t) = 0$ SATISFIES $y(2) = 0$.

SO THE SOL'N IS

$$\boxed{y(t) = 0}$$

TWO TYPES OF SOL'NS!

Observations:

A 1st order differential equation can have:

1. No Solution

2. Infinitely many solutions (one “parameter” or “free constant”, initial conditions not given)

3. A unique solutions (initial conditions given)

In a class on the theory of differential equations you would talk about ~~how~~ this is more detail (conditions on the differential equations in order for a solution to exist and be unique). Read 2.4 and ask me questions if you are interested in learning more.

Example: Find an *implicit* solution to

$$\frac{dy}{dx} = \frac{3x+1}{5y^4 - y}$$

with $y(2) = 1$.

$$\int (5y^4 - y)\, dy = \int (3x+1)\, dx$$

$$\boxed{y^5 - \frac{1}{2}y^2 = \frac{3}{2}x^2 + x + C_1}$$

HARD/IMPOSSIBLE TO NICELY SOLVE FOR y

$$y(2) = 1 \Rightarrow \begin{cases} x = 2 \\ y = 1 \end{cases}$$

$$(1)^5 - \frac{1}{2}(1)^2 = \frac{3}{2}(2)^2 + (2) + C_1$$

$$\Rightarrow \quad \frac{1}{2} = 6 + 2 + C_1$$

$$C_1 = \frac{1}{2} - 8 = -\frac{15}{2}$$

$$\boxed{y^5 - \frac{1}{2}y^2 = \frac{3}{2}x^2 + x - \frac{15}{2}}$$

Example: Find the general *explicit* solution to

$$2\frac{dy}{dx} = 3x^2(y^2-1)$$

$$\frac{2}{y^2-1} = \frac{2}{(y+1)(y-1)} = \frac{A}{y+1} + \frac{B}{y-1}$$

$$\Rightarrow \quad 2 = A(y-1) + B(y+1)$$

one way: $2 = (A+B)y + (-A+B)$

So $A+B=0$, $-A+B=2$

another: $y=1 \Rightarrow 2 = A(0) + B(2)$, $B=1$

$y=-1 \Rightarrow 2 = A(-2) + B(0)$, $A=-1$

RECALL: $\ln|a| - \ln|b| = \ln\left|\frac{a}{b}\right|$

$$\int \frac{2}{y^2-1}\,dy = \int 3x^2\,dx$$

$$\int \frac{-1}{y+1} + \frac{1}{y-1}\,dy = x^3 + C_1$$

$$-\ln|y+1| + \ln|y-1| = x^3 + C_1$$

$$\Rightarrow \quad \ln\left|\frac{y-1}{y+1}\right| = x^3 + C_1$$

$$\left|\frac{y-1}{y+1}\right| = e^{(x^3+C_1)}$$

$$\frac{y-1}{y+1} = \underbrace{\pm e^{C_1}}_{C} e^{x^3}$$

$$y-1 = Ce^{x^3}(y+1)$$

$$y-1 = Ce^{x^3}y + Ce^{x^3}$$

$$y - Ce^{x^3}y = 1 + Ce^{x^3}$$

$$y(1-Ce^{x^3}) = 1 + Ce^{x^3}$$

$$\boxed{y = \frac{1+Ce^{x^3}}{1-Ce^{x^3}}}$$

Example:

A town currently has 2100 people

- The birth/death rate is proportional to the population size with a relative growth rate of $k = 0.03$.
- In addition, 100 people/year are immigrating into the city from elsewhere.

Let $P(t)$ be the number of people in the city in t years from now.

Find $P(t)$.

RECALL: $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$

$$\frac{dP}{dt} = 0.03P + 100$$

$$\Rightarrow \int \frac{1}{0.03P+100}\,dP = \int 1\,dt$$

$$\frac{1}{0.03}\ln|0.03P+100| = t + C_1$$

$$\ln|0.03P+100| = 0.03t + C_2$$

$C_2 = 0.03C_1$

$$0.03P + 100 = \underbrace{\pm e^{C_2}}_{C_3} e^{0.03t}$$

$$0.03P = C_3 e^{0.03t} - 100$$

$$P = \frac{C_3}{0.03} e^{0.03t} - \frac{100}{0.03}$$

$$P(t) = Ce^{0.03t} - 3333.\bar{3}$$

$\frac{10000}{3}$

$$P(0) = 2100 \Rightarrow C - 3333.\bar{3} = 2100$$

$$C = 5433.\bar{3}$$

$$\boxed{P(t) = 5433.\bar{3}e^{0.03t} - 3333.\bar{3}}$$

NOT $P(0) = 2100$

$P(1) \approx 2265.47$

3% of 2100 is about 63

about 165 more people

Example:

Consider

$$\frac{dy}{dx} = 3x - y$$

This is NOT separable. It is "linear" and we will discuss a method on Wednesday for this type.

But if you leave this course, you may encounter a method called "change of variable" to "fix" a problem like this. Let's try one.

Assume I tell you to let $\overset{v(x)}{v} = 3x - \overset{y(x)}{y}$

Find

$$\frac{dv}{dx} = 3 - \frac{dy}{dx} = 3 - \underbrace{(3x - y)}_{v}$$

This new equation is separable!! Solve it, then rewrite your final answer in terms of y and x.

$$\frac{dv}{dx} = 3 - v$$

$$\Rightarrow \int \frac{1}{3 - v}\, dv = \int 1\, dx$$

$$\frac{1}{-1} \ln|3 - v| = x + C_1$$

$$\ln|3 - v| = -x - C_1$$

$$|3 - v| = e^{-x} e^{-C_1}$$

$$3 - v = \underbrace{\pm e^{-C_1}}_{C_2} e^{-x}$$

$$\boxed{v = 3 - C_2 e^{-x}}$$

$$3x - y = 3 - C e^{-x}$$

$$y = 3x - (3 - Ce^{-x})$$

$$\boxed{y = 3x - 3 + Ce^{-x}}$$

Check!